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HIGH INTERPOLATION ELEMENT ORDER IN A PETROV-GALERKIN METHOD FOR CFD PROBLEMS

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Abstract. *In this paper, a stable Petrov-Galerkin formulation for the compressible Euler equations is combined with a* **p***-adaptive remeshing refinement. The stability engendered by this formulation allows using the same spatial interpolation order in all the variables even in the presence of steep gradients. The result is an accurate and efficient scheme appropriated to solve problems presenting shocks and boundary-layers.*

Key-words: *p enrichment, Petrov-Galerkin, compressible fows*

1. INTRODUCTION

In the last years, with the advent of successful discretization methods and powerful computer technologies, computer simulation has been seen as a valuable approach for solving complicated transport problems such as compressible flows in aerospace applications and many environmental problems. For these problems numerical instabilities leading to globally spurious oscillations can occur if an inappropriate approximation for the convective term is done. Among the difficulties associated with numerical simulation of compressible flows is the representation of shocks and boundary-layers. In those regions of the fluid flow, the variables in the system vary strongly, making the computation very challenging. In that case, it is well known that the approximate solution obtained by using the standard Galerkin finite element is completely spoiled by spurious oscillations that are spread all over the computational domain. In order to overcome or, at least, to minimize those oscillations many methods have been designed and we should mention the Petrov-Galerkin models which modify the Galerkin's weighting functions by adding a perturbation term but keeping the consistency property in the sense that the exact solution satisfies the approximate problem. In the context of finite element methods a remarkable improvement in the development of consistent, stable and accurate methods for convection-dominated problems was provided by the SUPG Method (*Streamline Upwind Petrov-Galerkin Method*) (Brooks and Hughes, 1982) which has good stability and accuracy properties if the exact solution is regular and eliminates global pollutant effects for non-regular solutions. However, Gibbs' phenomena may still appear in the vicinity of sharp discontinuities such as shock waves. They can be completely prevented by using the CAU Method (*Consistent Approximate Upwind Method*) (Almeida and Galeão, 1993), which adds, in a consistent way, a non-linear perturbation that provides the control over the derivatives in the direction of the generalized approximate gradient. The stability enhancement allows the use of equal order interpolation in space for all the variables such that the extension to high-order interpolation elements is straightforward. Usually, higher than one interpolation orders are used only out of regions containing steep gradients because they produce or amplify instabilities in the approximate solution (de Cougny et al., 1994; Bey and Oden, 1993). As the stability is guaranteed owing to the CAU method, incorporating hierarchical element functions would enable *hp*-adaptive strategies which have known advantages over h-adaptivity schemes in many cases. Indeed, it was shown in (Almeida and Silva, 1997) that the use of p refinement improves the approximate solution in regions with discontinuities for the scalar convection-dominated convective diffusive problems.

In this work, a CAU type method (RVCAU), derived in (Almeida and Silva, 1997) for the scalar convection-diffusion problem is combined with a p-adaptivity scheme for solving the steady state solution of the bidimensional compressible Euler equations. To guide the adaptivity scheme, it is used an error indicator based on the norm used in (Hughes et al., 1987) for the convergence analysis of the SUPG formulation for linear time-dependent multidimensional advective-diffusive systems (Almeida and Galeão, 1996). The weak form is based on the time-dicontinuous Galerkin with piecewise constant interpolation used in time. With a single Newton step performed at each time, the steady-state solution is rapidly reached maching in time.

An outline of this paper is as follows. In section 2 the flow problem under consideration is described and the variational formulation for systems employing entropy variables is presented. In section 3 the p-adaptive scheme is discussed. In section 4 numerical experiments are conducted and the conclusions are drawn in section 5.

2. STATEMENT OF THE PROBLEM

In this work we are interested in finding the steady state solution of the bi-dimensional compressible Euler equations. In such nonlinear problems it is known that a good strategy is to get the steady state solution from the limit solution of a time-dependent problem. As we shall use this strategy here, we will consider first the following V-entropy variables (Hughes et al., 1986) formulation for the incompletely parabolic Navier-Stokes equations:

$$A_0 V_{,t} + \widetilde{A} \cdot \nabla V = \nabla \cdot \widetilde{K}\, \nabla V + \wp \tag{1}$$

where $A_0 = \frac{\partial U}{\partial V}$ and $U = (U_j)_{j=1}^m$ is the conservation variables vector, which in two dimensions is $U^t = [\rho, \rho u_1, ..., \rho u_d, \rho e]$. Here, ρ is the density; u_i is the velocity in $i^{\underline{th}}$ direction, $i = 1, ..., d$ ($d = 2$ and $m = 4$ for bidimensional problems); e is the total energy density. $\widetilde{A}_i = A_i A_0$, $\widetilde{K}_{ij} = K_{ij} A_0$, $\nabla^t = \left(Im \frac{\partial}{\partial x_1}, ..., Im \frac{\partial}{\partial x_d} \right)$ and Im is the mxm identity matrix. $F_{i,i} = F_{i,U} U_{,i} = A_i U_{,i}$; $F_i^\nu = K_{ij}^\nu U_{,j}$ and $F_i^h = K_{ij}^h U_{,j}$, where F_i is the Euler flux; F_i^ν is the viscous flux, F_i^h is the heat flux and $\wp$ is a source vector. In the following we consider the compressible Euler equations as a particular case of the Navier-Stokes equations when F_i^ν and F_i^h vanish. The usage of these (physical) entropy variables may be explained by the fact that the Galerkin formulation of the compressible Navier-Stokes equations based on the conservative variables lacks certain properties which are needed to establish stability proofs and convergence analysis (Shakib, 1985).

For the Euler equations let the approximated solution residual be

$$\mathcal{L}^h = A_0 V^h_{,t} + \widetilde{A} \cdot \nabla V^h \tag{2}$$

The CAU type method for (1) may now be formulated by using the time-discontinuous Galerkin method as the basis of our formulation. To this end, consider partitions $0 = t_0 < t_1 ... < t_n < t_{n+1} < ...$ of $\Re^+$ and denote by $I_n = (t_n, t_{n+1})$ the $n^{\underline{th}}$ time interval. The space-time integration domain is the "slab" $S_n = \Re^d$ x I_n with boundary $\overline{\Gamma} = \Gamma$ x I_n and denote by S_n^e de e^{th} element in S_n, $e = 1, ...(N_e)_n$, where $(N_e)_n$ is the total number of elements in S_n. For $n = 0, 1, 2...$ the space-time finite element space $\mathcal{U}_n^h$ consists of continuous piecewise polynomials on the slab S_n and may be discontinuous in time across the time levels t_n. The variational formulation consists of: Find $V^h \in S_n^h$ such that for $n = 0, 1, 2...$

$$\int_{S_n} \widehat{V}^h \cdot \ \mathcal{L}^h \ d\mathbf{x}dt + \sum_{e=1}^{(N_e)_n} \int_{S_n^e} (\ A_0 \widehat{V}^h + \widetilde{A} \cdot \nabla \widehat{V}^h \ ) \cdot \tau^e \ \mathcal{L}^h \ d\mathbf{x}dt +$$

$$\sum_{e=1}^{(N_e)_n} \int_{S_n^e} \nabla \widehat{V}^h \cdot [A_0] \, \nabla V^h \ \left(\frac{{\mathcal{L}^h}^t}{|\nabla V^h|^2_{A_0}} \ \tau_c^e \ \ \mathcal{L}^h \right) \ d\mathbf{x}dt \qquad + \tag{3}$$

$$\int_{\Re^d} \widehat{V}^h(t_n^+) \cdot \ A_0 \ \left(V^h(t_n^+) - V^h(t_n^-) \right) d\mathbf{x} = 0 \ , \qquad \forall \ \widehat{V}^h \in \mathcal{U}_n^h \ ,$$

where the last term represents the jump condition by which the information is propagated from one slab to the next. The term involving τ^e - the (m x m) matrix of intrinsic time scale

$$\tau^e = A_0^{-1} \left[\frac{\partial \xi_0}{\partial x_0} I_m + \left(\frac{\partial \xi_i}{\partial x_j} \frac{\partial \xi_i}{\partial x_k} \right) A_j A_k \right]^{-\frac{1}{2}} \tag{4}$$

as defined in (Shakib, 1985) - is the SUPG contribution. In (4), $x_0 = t$, x_i are the cartesian coordinates and ξ_i are the local element coordinates, $i = 1, 2$. Notice that the SUPG operator here is acting over the generalized characteristics. Such approach seems to be more appropriate when using the space-time formulation adopted in this work and provides more stability to the weak form in transient problems (Hughes et al., 1987) (in the next section, the definition of this intrinsic time scale function will be discussed as far as higher interpolation element orders are concerned). The discontinuity-capturing operator introduced by the CAU method corresponds to the third term in (3). $\tau_c^e = \frac{h^e}{2} \mathcal{U} \ \lambda^{-1} \mathcal{U}^t$ for the considered hyperbolic case and if a uniform mesh is used, where h^e is the characteristic length and λ and $\mathcal{U}$ are

$$\lambda^{\ 2} = \frac{\left| \mathcal{L}^h \right|^2_{A_0^{-1}}}{|\nabla V^h|^2_{A_0}} \ ; \quad \mathcal{U} = \frac{A_0^{-1} \mathcal{L}^h}{|\mathcal{L}^h|_{A_0^{-1}}} \tag{5}$$

Otherwise

$$\tau_c^e \ = \mathcal{U} \ \ \mu_c^{-1} \ \mathcal{U}^t \ , \tag{6}$$

with

$$\mu_c^2 = \lambda^2 \frac{\left|\nabla_\xi V^h\right|^2_{A_0}}{\left|\nabla V^h\right|^2_{A_0}} = \frac{\left|\mathcal{L}^h\right|^2_{A_0^{-1}}}{\left|\nabla V^h\right|^2_{A_0}} \frac{\left|\nabla_\xi V^h\right|^2_{A_0}}{\left|\nabla V^h\right|^2_{A_0}} \qquad \text{if} \quad \left|\nabla V^h\right|_{A_0} \neq 0$$

$$\mu_c = 0 \qquad \text{if} \quad \left|\nabla V^h\right|_{A_0} = 0 \ , \tag{7}$$

where $\left(\nabla_\xi V^h\right)^t = \left[V^h_{,i}\,\frac{\partial \xi_1}{\partial x_i},\ \ldots,\ V^h_{,i}\,\frac{\partial \xi_d}{\partial x_i}\right]$. Using these results in (3), the CAU term is written as

$$T_{CAU} = \sum_{e=1}^{(N_e)_n} \int_{S_n^e} \mu_c^{-1} \frac{\left|\mathcal{L}^h\right|^2_{A_0^{-1}}}{\left|\nabla V^h\right|^2_{A_0}} \nabla \hat{V}^{h^t} \left[A_0\right] \nabla V^h \ \ d\mathbf{x}dt \ . \tag{8}$$

In order to avoid the double effect when the approximate gradient direction coincides with the generalized SUPG direction, we should subtract the projection of the SUPG operator in this direction (see (Almeida and Silva, 1997)). Thus, τ_c^e can be redefined as

$$\tau_c^e = \max\left\{0,\ \mathcal{U}\,\mu_c^{-1}\mathcal{U}^t \ - \ \mathcal{U}\frac{\overline{\nabla} V^{h^t}\overline{A}\ \tau^e\ \mathcal{L}^h}{|\mathcal{L}^h|^2_{A_0^{-1}}}\mathcal{U}^t\right\} \ . \tag{9}$$

where $\overline{\nabla}^t\left(\cdot\right) = \left[I_m\frac{\partial(\cdot)}{\partial t}, I_m\frac{\partial(\cdot)}{\partial x_1}, \ldots, I_m\frac{\partial(\cdot)}{\partial x_d}\right]$ and $\overline{A}^t = \left[A_0, \widetilde{A}_1, \ldots, \widetilde{A}_d\right]$.

The resulting CAU term can be written now as

$$T_{CAU} =$$

$$\sum_{e=1}^{(N_e)_n} \int_{S_n^e} \max\left\{0,\ \mu_c^{-1}\frac{\left|\mathcal{L}^h\right|^2_{A_0^{-1}}}{|\nabla V^h|^2_{A_0}} - \frac{\overline{\nabla} V^{h^t}\overline{A}\ \tau^e\ \mathcal{L}^h}{|\nabla V^h|^2_{A_0}}\right\} \nabla \hat{V}^{h^t}\left[A_0\right]\nabla V^h\ d\mathbf{x}dt \ . \tag{10}$$

3. THE ADAPTIVE REFINEMENT METHOD

The adaptive scheme combined with the CAU method consists in performing the calculations beginning with an initial coarse mesh until the steady state is reached; then, the error is estimated and, if the prescribed accuracy has not been satisfied, a new mesh is generated and the calculations are carried out using as initial condition the values of the variables interpolated at the nodes on the new mesh. As our aim in this work is to check the ability of high interpolation orders in representing regions with discontinuities, only p-refinement scheme is performed.

The convergence analysis of the generalized SUPG formulation for linear time-dependent multidimensional advective-diffusive systems were performed in (Hughes et al., 1987) yielding uniform error estimates analogous to the scalar convection-diffusion problem. Indeed, the norm used in this case has the same features of its scalar counterpart. Then, it seems quite natural to have an error indicator in this way, even for the nonlinear compressible Euler equations. Thus, as first proposed in (Almeida and Galeão, 1996), for each element e, $e = 1, \ldots (N_e)_n$, the error indicator to be used in this paper is defined as:

$$\epsilon_e^2 = \int_{S_n^e} \left(\widetilde{A}\cdot\nabla^*\epsilon\right)\cdot\tau\ \left(\widetilde{A}\cdot\nabla^*\epsilon\right)d\mathbf{x} \tag{11}$$

where $\nabla^* \epsilon = \nabla^* V - \nabla V^h$. Here, $\nabla^* V \in [C^0(\Omega)]^m$ is an approximation of the exact gradient solution, obtained by nodal averaging of the discontinuous ∇V^h surrounding each node. The square of the global ϵ can be determined by summing all element contributions:

$$\epsilon = \left[\sum_{e=1}^{(N_e)_n} \epsilon_e^2 \right]^{\frac{1}{2}} \tag{12}$$

Here the adaptive strategy which seeks an *optimal mesh* in the sense that the error is equally distributed for all elements is adopted. The desired error is denoted by

$$\overline{\epsilon}_m = \overline{\eta} \left[\frac{\sum_{e=1}^{(N_e)_n} \left(\left\| V^h \right\|_e^2 + \epsilon_e^2 \right)}{N} \right]^{\frac{1}{2}} \tag{13}$$

where $\overline{\eta}$ is the specified maximum admissible percentage error. The term under the summation is an approximation for $\|V\|^2$ (V is the unknown exact solution) and $\|\cdot\|$ is determined applying the same definition used for the error indicator (11).

In this work, the 2-D hp Adaptive Package (2DhpAP) developed in (Demkowicz et al., 1992) is used. Its main features regarding p-adaptivity are as follows. During the p-refinement, the order of approximation for the element is increased by including polynomials of higher order in the element shape functions. This, sometimes called p-enrichment, may be done selectively for each of the element nodes as the order of approximation is associated with a node rather than the element. This is done by defining a master element on the basis of a right triangle with seven nodes: three vertices, three mid-side nodes and one middle node. Each of the mid-side and the middle nodes may have a separate order of approximation, denoted by $p_1, \ldots, p_4$, respectively. For each node the corresponding shape function is introduced, i.e., linear shape functions for the three vertices, higher order shape functions for mid-side nodes and for middle nodes. It should be remarked that these mid-side nodes functions corresponding to one side vanish along the two remaining sides. When complemented with the linear shape functions corresponding to the side endpoints, they span polynomials of order p_i. The middle node shape functions vanish along the whole element boundary and the element shape functions, altogether, span polynomials of order $p = \min(p_1, p_2, p_3, p_4)$. Notice that the shape functions corresponding to one node are not required to be hierarchical in the sense that the shape functions of order $p+1$ are constructed by adding additional $(p+1)$-order polynomials to shape functions of order p.

The main ingredient in combining p-adaptivity with the stabilized CAU method is the choice of the matrix of intrinsic time scale: a proper evaluation of this parameter is crucial to guarantee the desired accuracy. An interesting result presented in (Almeida and Silva, 1997) for the scalar convection-diffusion problem shows that the upwind parameter is very sensitive to the interpolation order of the element. This means that its definition must depend on the interpolation order in each element. Many authors have defined such dependence either heuristically (Shakib, 1985; Zienkiewicz and Taylor, 1988) or by studing exact discrete solutions for the scalar convection-diffusion one-dimensional problem using the SUPG with quadratic and hierarchical elements of order equal to two (Codina et al., 1992).

For higher interpolation orders, it was numerically shown in (Almeida and Silva, 1997) that this function should be reduced with the growth of the interpolation order of the element about $(p)^{-1}$. With this choice, either the regular solutions are acuratelly resolved

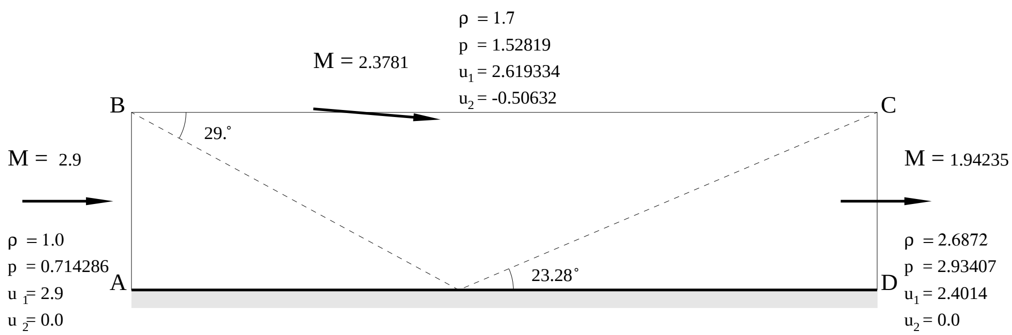

Figura 1: Problem statement: reflection shock against a wall

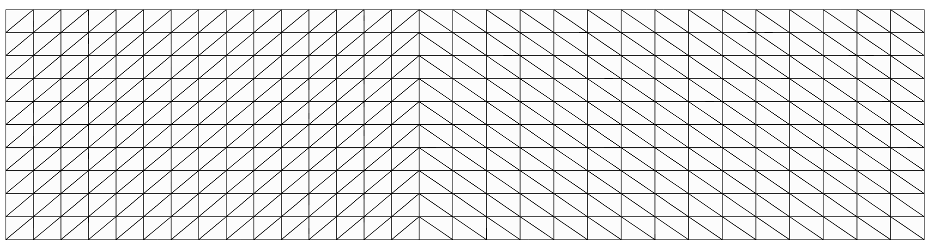

Figura 2: First mesh (341 nodes)

or the stability is kept for non regular solutions. Indeed, the use of higher-order elements in regions with steep gradients successfully improve their representation. Here, this idea is extended to the system of Euler equations. Both the matrix τ^e and the scalar τ_c^e are divided by $p = \min(p_1, p_2, p_3, p_4)$. The results, that will be shown next section, are promising.

4. NUMERICAL RESULTS

In this section some numerical results obtained by applying the proposed methodology on the solution of a compressible Euler flow problem of the reflection shock against a wall are shown. This inviscid problem, presented in Fig. 1, deals with three regions where the supersonic flow is constant which are separeted by shocks. The computational domain is a rectangle with 4.1 and 1 of length in the x and y directions, respectively. Along the inflow AC all variables are fixed; along AD a slip boundary condition is enforced and along DC all variables are left free.

Beginning with the mesh depicted in Fig. 2, sequential p-refinements are performed until refinement level 4 is reached. This means that the maximum possible interpolation order is $p = 4$. In Figs. 3 - 4 and 5 - 6 the meshes and the density solutions for the last p-refinement are shown, either introducing or not the reduction over the upwind terms, respectively. The color scale in meshes graphics corresponds to polynomial degrees p (darker colors corresponds to greater p). The overlapping colors mean the enrichment of polynomial interfaces to mantain continuity of shape functions across interelement boundaries.

The evolution of the adaptivity refinement is shown in Figs. 7 and 8 where the density line plots at $y = 0.2$ are depicted together with the exact solution when the

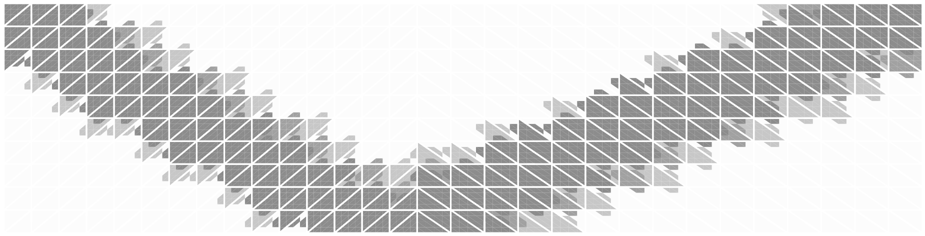

Figura 3: Final mesh without reduction (341 vertices, 448 mid-side nodes and 204 middle nodes)

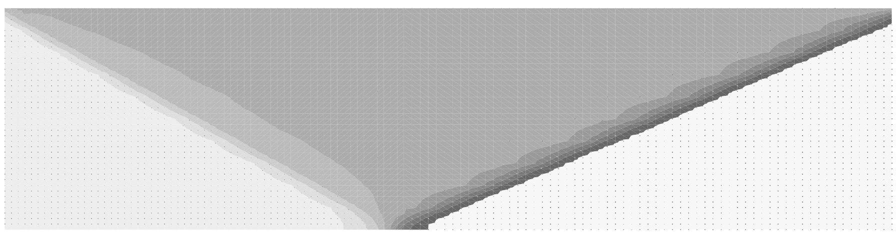

Figura 4: Final solution without reduction

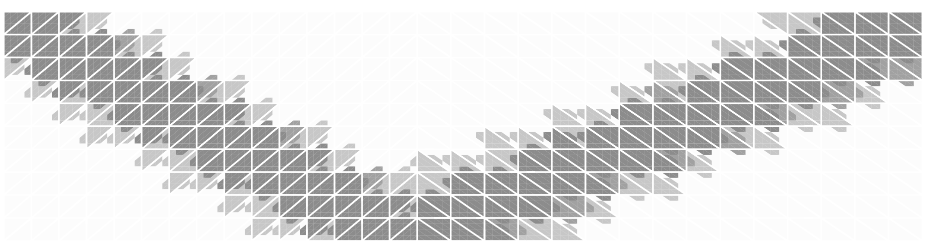

Figura 5: Final mesh with reduction (341 vertices, 445 mid-side nodes and 180 middle nodes)

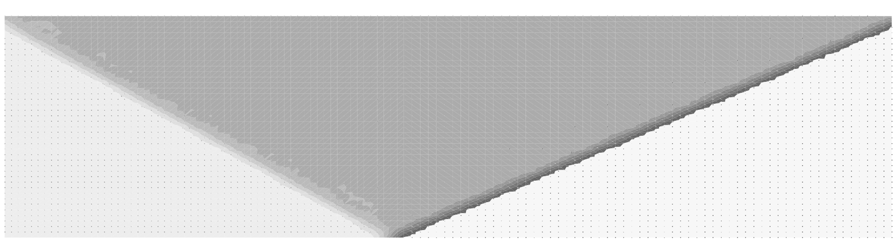

Figura 6: Final solution with reduction

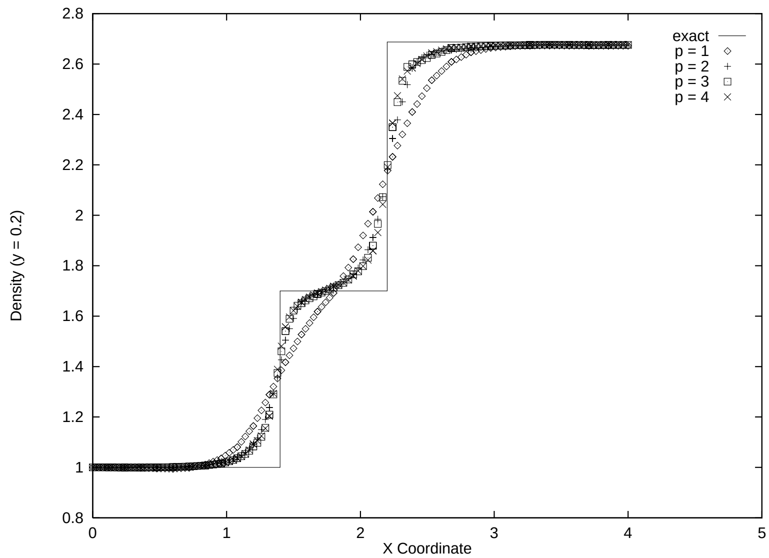

Figura 7: Density line plots at $y = 0.2$ (no reduction)

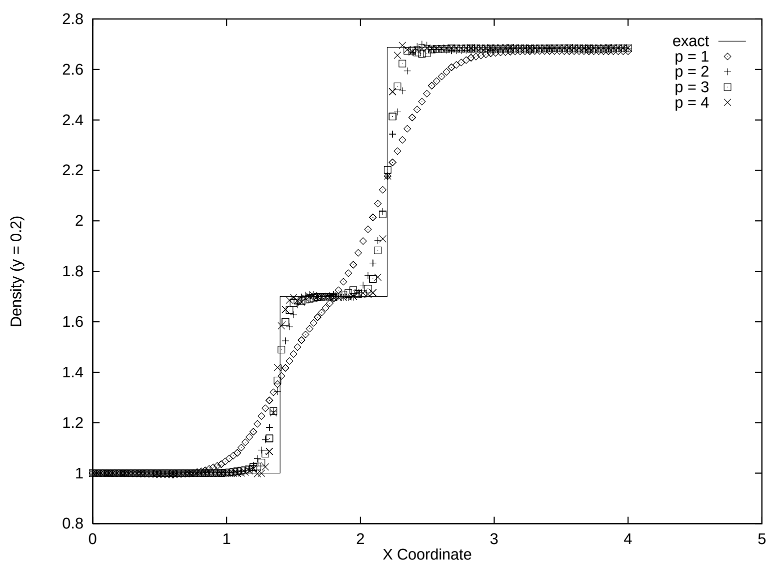

Figura 8: Density line plots at $y = 0.2$ (with reduction)

upwind reduction is not introduced and when it is applied, respectively. These cases are compared in Fig. 9 for the last refinement step. For the reduction case, the density profile at $y = 0.2$ is also shown in Fig. 10 where the gray scale corresponds to polynomial degrees p. These results point out the improvement in representing shocks when decreasing the upwind terms with the increase of the interpolation order. Notice the improvement of the shock representation with the increase of the degree p.

5. CONCLUSIONS

In this paper a stable Petrov-Galerkin method for solving the compressible Euler equations written in entropy variables is combined with a p-adaptivity scheme. The good stabilisation properties allow the use of higher interpolation elements in regions with shocks. However, it is shown that the shock representation can be improved by modifying

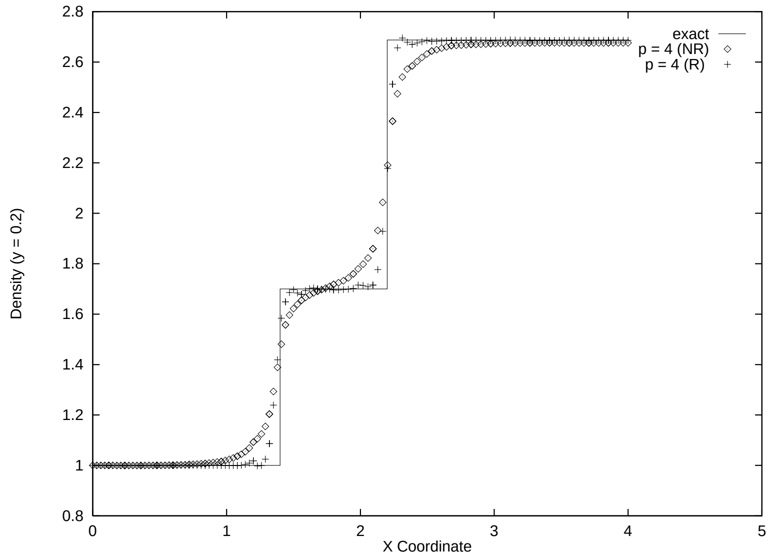

Figura 9: Comparison between the two strategies

the upwind terms with the growth of the element interpolation order. The numerical results showed that this model is very stable, providing accurate approximate solutions near shocks free of spurious oscillations.

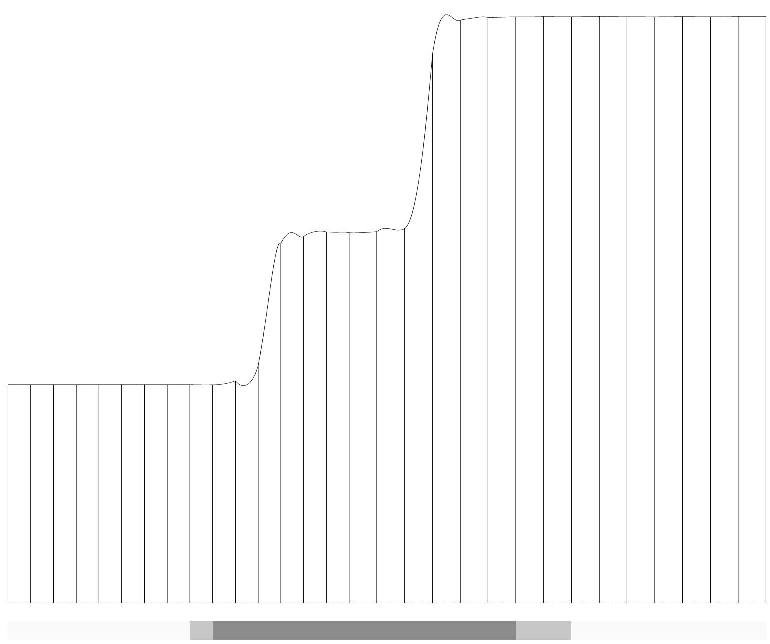

Figura 10: Density profile at $y = 0.2$ (with reduction)

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